

Entanglement Entropy and Quantum Phase Transition in the $O(N)$ σ -model

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Abstract

We investigate how entanglement entropy behaves in a system with a quantum phase transition. We study the σ -model which has an $O(N)$ symmetry when the mass squared parameter μ^2 is positive, and when μ^2 is negative, this symmetry is broken spontaneously. The area law and the leading divergence of entanglement entropy are preserved in both the symmetric and the broken phases. In 3+1 dimensions, the spontaneous symmetry breaking changes the subleading divergence from a log to log squared, while in 2+1 dimensions the subleading divergent structure is unchanged. At the leading order of the coupling constant expansion, the entanglement entropy reaches its local maximum with a cusp at the quantum phase transition point $\mu^2 = 0$ and decreases while $|\mu^2|$ is increased. We also find novel scaling behavior of the entanglement entropy near the transition point.

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I. INTRODUCTION

Entanglement[1] is an intriguing feature of Quantum Mechanics. It has become increasingly important in quantum information[2], condensed matter[3–6], and string theory[7], and even in the interpretation of the black hole entropy[8]. For example, for the condensed matter systems, the ground states of the conventional superconductors[9, 10] and the fractional quantum Hall effect[11] are both entangled states. Such systems may undergo quantum phase transition at zero temperature by tuning their physical parameters. Quantum phase transition[12] is fundamentally different from the conventional thermal phase transition occurred at non-zero temperature, in the sense that it involves a qualitative change in the ground state of a quantum system. We know the thermal entropy characterizes the thermal phase transition. Similarly, the entanglement entropy, which measures the level of the entanglement between a certain region and its complement, can probe the quantum phase transitions, as it usually reaches a cusped maximum at the transition point[3, 6].

There appear universal properties at the quantum phase transition due to the emergent conformal symmetry. This had been demonstrated in the scaling behavior of the entanglement entropy in the 1+1 dimensional spin chain models [4, 5]. In stead of the discrete systems, in this paper we will study the entanglement entropy of the quantum phase transition in the field theory regime. The tractable model we focus on is the $O(N)$ σ -model in 2+1 and 3+1 dimensions. The symmetry breaking of $O(N)$ model at finite temperature had been investigated in [13]. We will apply the field theory technique to analyze the effect of quantum phase transition of the $O(N)$ σ -model due to the symmetry breaking on the entanglement entropy. It would be interesting if the analytic expression for the universal scaling behavior near the transition point can be unveiled.

In the $O(N)$ σ -model, the symmetry broken phase is characterized by the emergence of a non-trivial order, which is a non-trivial scalar field VEV that breaks the $O(N)$ spontaneously. In this phase, there exist the massless Goldstone bosons together with the massive scalar. Besides such spontaneous symmetry breaking (SSB) driven quantum phase transition, there are also quantum systems whose phase transitions do not involve any order parameters. One example is the topologically ordered phases [14]. In these cases, the entanglement entropy remains an important quantity identifying the topological order [15, 16]. Such systems, however, are beyond the scope of this paper.

The entanglement entropy (S_{ent}) between a subregion A and its complement \bar{A} had been calculated for the scalar fields in conformal field theory[17–22] and also in the non-conformal cases[18, 23]. The leading behavior of S_{ent} exhibits the area law[24] and the power law divergence, $S \sim \Lambda^{d-1} A_{\perp}$, where A_{\perp} is the area of the $d-1$ dimensional boundary of A , with d denoting the number of spatial dimensions, and Λ is the momentum cutoff. As for the subleading part in the non-conformal field theory, [23] demonstrates that, in terms of the bare mass and coupling constants, there exist terms proportional to $\lambda A_{\perp} \Lambda^2$ arising from the quartic interaction $-\frac{\lambda}{4!} \phi^4$, and terms proportional to $g^2 A_{\perp} \ln \Lambda^2$ from the cubic interaction $-\frac{g}{3!} \phi^3$. They are contributed by the two-loop quantum corrections. However, both can be absorbed into the mass renormalization of the scalar field ϕ , such that the subleading term of S_{ent} in 3+1 dimensions becomes $\frac{1}{48\pi} m_r^2 \ln m_r^2$, where m_r is the renormalized mass of the scalar field absorbing the cutoff dependence.

In this paper, the $O(N)$ σ -model with N species of the scalar fields lives in a bipartite system with A, \bar{A} each occupying the infinite half-space. The SSB is achieved by tuning the mass squared μ^2 of the scalar fields from positive to negative, and the system undergoes quantum phase transition as the vacuum states change. This is a perturbatively calculable system due to weak coupling, and we present our result up to two-loop corrections. While in the $O(N)$ symmetric phase this model has simple quartic interactions $\frac{\lambda}{4} \left[\sum_{i=1}^N (\phi^i)^2 \right]^2$, in the symmetry broken phase the cubic interactions $\frac{1}{\sqrt{2}} g m_{\sigma} \left(\sum_{i=1}^{N-1} (\pi^i)^2 \sigma + \sigma^3 \right)$ emerge besides the quartic ones, where σ is the massive mode, π^i 's are the Goldstone bosons, $g = \sqrt{\lambda}$, and $m_{\sigma} = \sqrt{-2\mu^2}$.

We are interested in the leading and subleading UV divergences of the entanglement entropy in this paper. In contrast to [23], we employ the renormalized mass, coupling constant and fields at the tree level, and introduce the counter terms to cancel the quantum loop corrections. The counter terms in the symmetry broken phase inherit from those in the $O(N)$ symmetric phase. We find that, in 3+1 and 2+1 dimensions, the area law and the leading divergence structure are preserved in both phases, as expected. The entanglement entropy reaches its maximum with a cusp on the phase boundary at $\mu^2 = 0$. While $|\mu^2|$ is tuned up, the entanglement entropy reduces in both phases as the correlation length reduces away from the quantum critical point.

According to our renormalization scheme, we find that, up to two-loop expansions, the divergence structure of the entanglement entropy of the σ model in the $O(N)$ symmetric

and the broken phase can be summarized by the following expressions in 3+1 dimensions:

$$S_{\text{ent.}}(\mu^2, \lambda) = \frac{A_{\perp}^{(2)} \Lambda^2}{48\pi} N \left\{ \ln 4 + a\tilde{\lambda} \left(\frac{|\mu^2|}{\Lambda^2} \right) \left[\ln \left(\frac{|\mu^2|}{\Lambda^2} \right) \right]^2 + (b + c\tilde{\lambda}) \left(\frac{|\mu^2|}{\Lambda^2} \right) \ln \left(\frac{|\mu^2|}{\Lambda^2} \right) + O\left(\tilde{\lambda}^2, \frac{|\mu^2|}{\Lambda^2}\right) \right\}, \quad (1)$$

where the coefficients are

	$O(N)$ symm.	SSB
a	0	$-\frac{3}{(4\pi N)^2}(N+2)$
b	1	$\frac{2}{N}$
c	0	$-\frac{2}{(4\pi N)^2}\{9\sqrt{5}\ln(\frac{3+\sqrt{5}}{2}) + (6\ln 2 - 2) + (3\ln 2 + 2)N\}$

While in 2+1 dimensions the entanglement entropy reads

$$S_{\text{ent.}}(\mu^2, \lambda) = \frac{A_{\perp}^{(1)} \Lambda}{24\pi} N \left\{ (\ln 4 + \pi) + \left(a\frac{\tilde{\lambda}}{\Lambda} + b\sqrt{\frac{|\mu^2|}{\Lambda^2}} \right) + O(\tilde{\lambda}^2, \frac{|\mu^2|}{\Lambda^2}) \right\}, \quad (2)$$

with

	$O(N)$ symm.	SSB
a	0	$-\frac{1.645+1.805(N-1)}{8\pi N^2}$
b	-2π	$-\frac{2\sqrt{2}\pi}{N}$

In these expressions, N is the number of species of the scalar fields, and $\tilde{\lambda} = \lambda/N$. Note that μ , m_{σ} and λ all stand for the renormalized parameters. The area law is clearly observed, with $A_{\perp}^{(2)}$ and $A_{\perp}^{(1)}$ denoting the area of the 2- and 1-dimensional boundary surface of A in 3+1 and 2+1 dimensions, respectively. There is also the cutoff dependence in the leading divergence.

In 3+1 dimensions, it is found that in the $O(N)$ symmetric phase, the subleading term exhibits log divergence, while in the symmetry broken phase the subleading divergence becomes log squared. The latter arises from the remnant of cancellation between the two-loop expansions of the cubic interactions which emerge due to the SSB, and the counter terms. The quartic interactions still give rise to the log divergence and become sub-subleading. In the symmetry broken phase in 2+1 dimensions, the additional term $\frac{\tilde{\lambda}}{\Lambda}$ shows up in the subleading part, without altering the divergence structure. (1) and (2) show that the mass dependence appears at the highest subleading terms. The scaling behavior of the entanglement entropy thus can be spelled out from these analytic expressions. (See Section V for details.)

The structure of this paper is organized as follows. We first review the fundamentals of the entanglement entropy and the replica method in quantum field theory in Sec. II. Sec. III presents the entanglement entropy of $O(N)$ σ -model in the $O(N)$ symmetric phase, up to two-loop perturbations. In Sec. IV, analogous calculation is carried out in the broken phase. Sec. V presents the numerical results of the entanglement entropy versus μ^2 in both phases, in $3 + 1$ and $2 + 1$ dimensions. Sec. VI. discusses and concludes our work, and presents potential future applications.

II. ENTANGLEMENT ENTROPY AND REPLICA METHOD

The thermal entropy indicates the level of disorder of a system. In the quantum case, the thermal entropy is given by the von Neumann entropy

$$S = \text{Tr}[\rho \ln \rho], \quad (3)$$

where ρ is the density matrix which is normalized to $\text{Tr}\rho = 1$. In the diagonalized basis, the von Neumann entropy reads $S = -\sum_i p_i \ln p_i$, where p_i is the probability for each microstates being occupied. By postulating the occupation of any microstate is equally probable, (3) is equivalent to the statistical definition of the entropy $S \sim \ln \Omega$ up to the Boltzmann constant, reflecting the total number of accessible microstates in a quantum system of microcanonical ensemble.

Consider a bipartite system S in a pure state and composed of subsystems A and \bar{A} , where the degrees of freedom in A , \bar{A} are entangled in some way. If one is forbidden to access \bar{A} , then for such an observer, A appears in a mixed state, with a reduced density matrix given by

$$\rho_A = \text{Tr}_{\bar{A}}\rho, \quad (4)$$

where \bar{A} is traced out. The information regarding the entanglement is encoded in ρ_A . As a result, the level of entanglement between A and \bar{A} is described by the entanglement entropy (EE), which is defined by

$$S_{\text{ent.}} = -\text{Tr}_A[\rho_A \ln \rho_A]. \quad (5)$$

Since the vacuum wave-function of \bar{A} is buried in the excited wave-functions of the “mix-state” subsystem A described by ρ_A , the expectation value of a local operator can be com-

puted by

$$\langle 0 | \mathcal{O}_A | 0 \rangle = \frac{\text{Tr}[\rho_A \mathcal{O}_A]}{\text{Tr}[\rho_A]}. \quad (6)$$

One example of such set-up is the black holes. Suppose that the whole spacetime is in a pure state, but we are unable to access the region inside the event horizon. Therefore the black hole appears thermal to an outside observer due to the entanglement between the two regions separated by the horizon, and so the entropy arise. This is one interpretation of the black hole entropy.

In this paper, we will consider the case that the system S contains the whole space, while the subsystems A and \bar{A} each occupies the infinite half-space, divided by a codimension 2 (with respect to the whole spacetime) surface. We follow the convention in [23], denoting the time t and radial coordinate x_{\parallel} as the *longitudinal directions*, as they are relevant in our field theory calculation, while the *transverse directions* indicate the dimensions of the surface enclosing the subsystem A .

In order to calculate the entanglement entropy, we take the generic scalar field theory as an example and review the replica method in the following. The entanglement entropy can be calculated by the following trick[18, 19, 23]

$$S_{\text{ent.}} = - \left. \frac{\partial}{\partial n} \right|_{n \rightarrow 1} \ln \text{Tr}[\rho_A^n] = -\text{Tr}[\rho_A \ln \rho_A], \quad (7)$$

where the trace is taken within A implicitly. As $n \rightarrow 1$, we can take $n = 1 + \epsilon$ and expand $\ln \text{Tr}[\rho_A^n]$ in ϵ for small ϵ . Then the entanglement entropy can be spelled out from the $O(\epsilon)$ term.

To calculate (7), we first notice that the elements of the reduced density matrix ρ_A can be expressed in the path integral formalism,

$$\langle \varphi_A | \rho_A | \varphi'_A \rangle = \int \mathcal{D}\phi \delta[\phi_A(\tau = 0^+) - \varphi_A] \delta[\phi_A(\tau = 0^-) - \varphi'_A] e^{-S_E[\phi = \phi_A \oplus \phi_{\bar{A}}]}, \quad (8)$$

where S_E is the action of ϕ over the whole Euclidean space with imaginary time τ . $\phi_A, \phi_{\bar{A}}$ are the scalar fields taking values in A, \bar{A} respectively. The field bases $|\varphi_A\rangle$ and $|\varphi'_A\rangle$ are states in A at certain time. In this expression, \bar{A} region is traced out. Since taking trace amounts to identifying the Euclidean time of the initial and the final states, (8) implies that ρ_A is computed on a manifold where \bar{A} is compactified (in τ direction) to a cylinder while A is left open. When we had identified $\phi(\tau = -\infty) = \phi(\tau = \infty)$, the matrix element of ρ_A^n is computed on a manifold on which \bar{A} consists of n cylinders on top of each other while

A becomes a n -sheeted spacetime manifold. Taking trace of ρ_A^n then joins the first sheet with the last for A , compactifying it into a cone with a total angle $2n\pi$ (or an excess angle $\delta = 2(n-1)\pi$), where $n \geq 1$. See e.g. Fig. 1 in [23] or Fig.'s 1 and 2 in [19] for the pictorial realization. As a result, it is natural to define the trace of ρ_A^n by

$$\ln \text{Tr}[\rho_A^n] = \ln \left(\frac{Z_n}{Z_1^n} \right), \quad (9)$$

where Z_n denotes the partition function of the field theory on the n -sheet manifold. ($n = 1$ reduces to the case on the ordinary Euclidean space.) The normalization by Z_1^n is due to the requirement that $\text{Tr}[\rho_A^n]|_{n \rightarrow 1} = 1$.

To summarize, using the replica trick, the entanglement entropy is calculated by

$$S_{\text{ent.}} = - \frac{\partial}{\partial n} [\ln Z_n - n \ln Z_1] \Big|_{n \rightarrow 1} = - \frac{1}{\epsilon} [\ln Z_n - n \ln Z_1]. \quad (10)$$

For $n > 1$, the replication of sheets takes place in the Euclidean time coordinate, which belong to the longitudinal directions, while the transverse directions remain ordinary Euclidean. We will adopt polar coordinates for the longitudinal part of the spacetime,

$$(\tau, x_{\parallel}) = (r \sin \frac{\theta}{n}, r \cos \frac{\theta}{n}), \quad (11)$$

where $r \in (0, +\infty)$ and θ is periodical with $2\pi n$. Thus the partition function on the n -sheet manifold is written down as

$$Z_n = \int \mathcal{D}\phi \exp \left[- \int d^{d_{\perp}} x_{\perp} \int_0^{\infty} r dr \int_0^{2\pi n} d\theta \mathcal{L}_{\text{E}}[\phi(r, \theta, x_{\perp})] \right]. \quad (12)$$

Such expression is valid only for the total spacetime dimensions $d + 1 > 2$.

Since the partition function $Z = \det(\partial^2 + \mu^2)$ is interpreted as the vacuum energy in quantum field theory, and the entanglement entropy of the is obtained by (10), $S_{\text{ent.}}$ can be interpreted as the derivative of the correction to the vacuum energy due to the cone with respect to the conical deficit angle. This notion will be more transparent as we calculate the free field entanglement entropy in the $O(N)$ symmetric phase in next section.

III. PERTURBATION EXPANSION OF THE $O(N)$ σ -MODEL IN THE SYMMETRIC PHASE

The Euclidean Lagrangian of $3 + 1$ dimensional $O(N)$ model is given by

$$\mathcal{L}_{\text{E}} = \sum_{i=1}^N \left[\frac{1}{2} (\partial \phi^i)^2 + \frac{1}{2} \mu^2 (\phi^i)^2 \right] + \frac{\lambda}{4} \left[\sum_{i=1}^N (\phi^i)^2 \right]^2 + \mathcal{L}_{\text{c.t.}}, \quad (13)$$

which has N species of scalar fields with the same mass μ , admitting $O(N)$ symmetry and quartic interactions. $\mathcal{L}_{c.t.}$ is the counter terms to cancel the loop corrections.

Since in this paper we use the renormalized mass μ and the renormalized coupling constant λ in the tree level action, the partition function on n -sheet manifold can be expanded with respect to λ by,

$$\ln Z_n = \ln Z_{n,0} + \sum_{j=1}^{\infty} \frac{(-\lambda)^j}{4^j j!} \int \left(\prod_{k=1}^j d^{d_{\perp}} x_{k\perp} \right) \int_n \left(\prod_{k=1}^j d^2 x_{k\parallel} \right) \left\{ \left\langle \left[\sum_{i=1}^N \phi^i(x_1)^2 \right]^2 \dots \left[\sum_{i=1}^N \phi^i(x_j)^2 \right]^2 \right\rangle_0 + \text{counter terms} \right\}, \quad (14)$$

where $\ln Z_{n,0}$ is the $O(\lambda^0)$ free field part. The counter terms are introduced to cancel the divergence from the perturbative corrections of loops, such that the renormalized μ and λ receives no further quantum corrections. \int_n is the integral over the 2-dimensional n -sheet manifold $\int_0^{\infty} r dr \int_0^{2\pi n} d\theta$.

In the following we calculate the entanglement entropy up to lowest-order corrections (in this case, $O(\lambda)$ bubble diagrams). The free field contribution is computed by the following method[23]. First, one notices that

$$\frac{\partial}{\partial \mu^2} \ln Z_{n,0} = -\frac{1}{2} \int_n d^{d+1} x G_n(x, x), \quad (15)$$

where $G_n(x, x')$ is the Green's function of the free scalars on the n -sheet Riemann surface, satisfying $(-\nabla^2 + \mu^2) G_n(x, x') = \delta^{d+1}(x - x')$. The expression for the Green's function on n -sheet Riemann surface $G_n(x, x')$ is very complicated; see [18] and [23] for details. As x' coincides with x , however, $G_n(x, x)$ can be decomposed into

$$G_n(x, x) = G_1(0) + f_n(r). \quad (16)$$

$G_1(0) = G_1(|x - x|)$ is the (divergent) Green's function on the Euclidean flat space, which admits translational invariance, and $f_n(r)$ represents the correction to the one-loop vacuum bubble due to the conical singularity [23]:

$$\begin{aligned} f_n(r) &= \frac{1}{2\pi n} \frac{1 - n^2}{6n} \int \frac{d^{d_{\perp}} p_{\perp}}{(2\pi)^{d_{\perp}}} K_0^2(\sqrt{\mu^2 + p_{\perp}^2} r) + \dots, \\ &\stackrel{n=1+\epsilon}{=} -\frac{\epsilon}{6\pi} \int \frac{d^{d_{\perp}} p_{\perp}}{(2\pi)^{d_{\perp}}} K_0^2(\sqrt{\mu^2 + p_{\perp}^2} r) + O(\epsilon^2) + \dots, \end{aligned} \quad (17)$$

with

$$\int_0^{\infty} dy y K_0^2(y) = \frac{1}{2} \quad \Rightarrow \quad \int_0^{\infty} r dr K_0^2(\sqrt{\mu^2 + p_{\perp}^2} r) = \frac{1}{2} \frac{1}{\mu^2 + p_{\perp}^2}, \quad (18)$$

where K_0 is the modified Bessel function of the second kind K_ν with $\nu = 0$, and \dots denotes the finite subleading terms. Since f_n is finite for $r > 0$ and decays exponentially at $r \rightarrow \infty$, one can see that f_n vanishes for $n \rightarrow 1$ (i.e. $\epsilon \rightarrow 0$), as expected.

Now we can make use of (15) and the subsequent approximation of $G_n(x, x)$ in (17) and (18) to calculate the free fields contribution to the entanglement entropy in the $O(N)$ symmetric phase from (10):

$$\begin{aligned} S_{\text{ent.}}^{\text{free}}(\mu^2) &= -\frac{1}{\epsilon} (\ln Z_{1+\epsilon,0} - (1+\epsilon) \ln Z_{1,0}) \\ &= \frac{N}{\epsilon} \int_{\infty}^{\mu^2} \frac{dm^2}{2} \int d^{d_\perp} x_\perp \left\{ \int_{1+\epsilon} d^2 x_\parallel G_n(x, x) - (1+\epsilon) \int_1 d^2 x_\parallel G_1(x, x) \right\}. \end{aligned}$$

The reason for integrating the mass squared parameter from ∞ to μ^2 is because we expect the entanglement entropy to vanish at $\mu^2 = \infty$, due to vanishing correlation length $\xi \sim \mu^{-1}$. We can further decompose the integration range into $\int_{\infty}^{\mu^2} dm^2 = \left(\int_{\infty}^0 + \int_0^{\mu^2} \right) dm^2$,

$$S_{\text{ent.}}^{\text{free}}(\mu^2) = \frac{N}{\epsilon} \left(\int_{\infty}^0 + \int_0^{\mu^2} \right) \frac{dm^2}{2} \int d^{d_\perp} x_\perp \int_{1+\epsilon} d^2 x_\parallel f_{1+\epsilon}(r) \quad (19)$$

$$\begin{aligned} &= -A_\perp \frac{N}{12} \int \frac{d^{d_\perp} p_\perp}{(2\pi)^{d_\perp}} \left[\int_{\infty}^0 \frac{dm^2}{m^2 + p_\perp^2} + \int_0^{\mu^2} \frac{dm^2}{m^2 + p_\perp^2} \right] \\ &= -A_\perp \frac{N}{12} \int \frac{d^{d_\perp} p_\perp}{(2\pi)^{d_\perp}} \left[\ln \frac{p_\perp^2}{\Lambda^2 + p_\perp^2} + \ln \frac{\mu^2 + p_\perp^2}{p_\perp^2} \right] \\ &= -A_\perp \frac{N}{12} \int \frac{d^{d_\perp} p_\perp}{(2\pi)^{d_\perp}} \ln \frac{\mu^2 + p_\perp^2}{\Lambda^2 + p_\perp^2} \quad (20) \end{aligned}$$

where $A_\perp = \int d^{d_\perp} x_\perp$ is the total area of the transverse space separating A and \bar{A} , and Λ is the mass scale (or momentum) cutoff in the integration over dm^2 . In the above calculation, $\int_n d^2 x_\parallel G_1(0)$ and $n \int_1 d^2 x_\parallel G_1(0)$ cancel out exactly.

In (20), the leading divergence $\sim \Lambda^{d-1}$ comes from $\int_{\infty}^0 \frac{dm^2}{2} \int d^{d_\perp} x_\perp \int_{1+\epsilon} d^2 x_\parallel f_{1+\epsilon}(r)$. This can be seen from the $(K_0)^2$ integral in (17) in e.g. 3+1 dimensions:

$$\frac{1}{\epsilon} r^2 f_n(r) \sim - \int dp_\perp p_\perp K_0^2(\sqrt{\mu^2 + p_\perp^2} r) = \frac{\mu^2}{2} (K_0(\mu r)^2 - K_1(\mu r)^2) \quad (21)$$

$$\sim \begin{cases} -\frac{1}{2} + \frac{(\mu r)^2}{2} [(\ln \mu r)^2 + \dots] + O(\mu^3 r^3), & r \rightarrow 0 \\ -e^{2\mu r}, & r \rightarrow \infty \end{cases} \quad (22)$$

as displayed in Fig. 1. Such behavior means, as the location of the quantum bubble is far away from the tip of the cone, the effect of the conical singularity is exponentially suppressed,

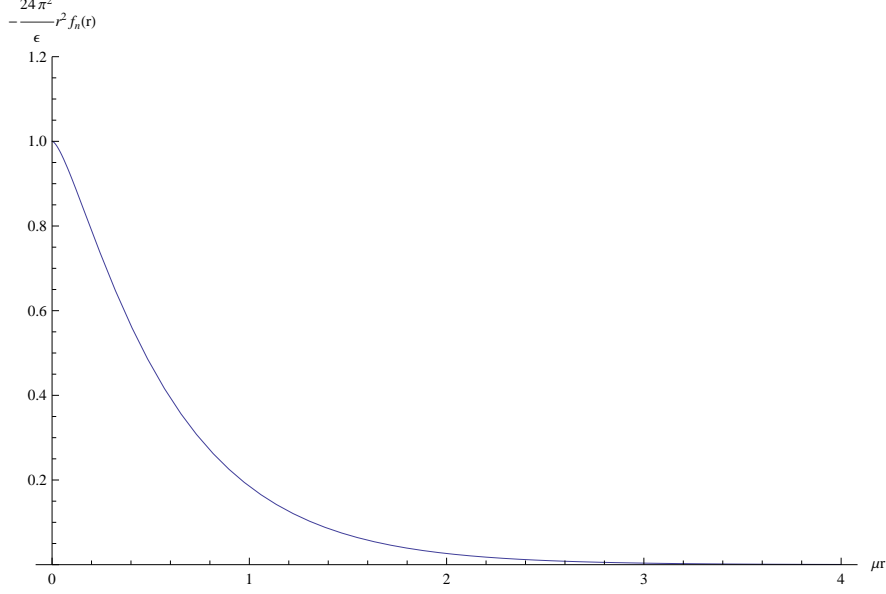


Figure 1: Behavior of $f_n(r)$, giving rise to the subleading divergence in $S_{\text{ent.}}^{\text{free}}(\mu^2)$.

and the bubble sees a flat Euclidean space. While the quantum bubble is very close to the conical point, it is $\frac{1}{r^2}$ divergent. Then plug (21) into (19), one finds that

$$\begin{aligned} S_{\text{ent.}}^{\text{free}}|_{\text{leading}} &\sim \int r dr \int_{\infty}^0 dm^2 \frac{m^2}{2} (K_0(\mu r)^2 - K_1(\mu r)^2) \\ &= \int r dr \frac{-1}{3r^4} \propto \Lambda^2 \end{aligned} \quad (23)$$

On the other hand, the subleading contribution $\sim \ln \Lambda$ to the entanglement entropy comes from the integral $\int_0^{\mu^2} \frac{dm^2}{2} \int d^d x_{\perp} \int_{1+\epsilon} d^2 x_{\parallel} f_{1+\epsilon}(r)$. It is straightforward to check by substituting the leading term in (22) at small r into (19), and then going through the calculation in (23).

Since the calculation of $S_{\text{ent.}}^{\text{free}}$ in (19) stems from (10), where f_n is the correction to the one-loop vacuum bubble due to the cone, it would be more clear here to comprehend the interpretation of the entanglement entropy as the derivative of the correction to the vacuum energy due to the cone with respect to the conical deficit angle, as mentioned in the end of the previous section.

The perturbation at $O(\lambda)$ level is

$$\begin{aligned} \ln Z_{n,1} &= -\frac{\lambda}{4} \int d^d x_{\perp} \int_n d^2 x_{\parallel} \left\langle \left[\sum_{i=1}^N \phi^i(x)^2 \right]^2 \right\rangle_0 \\ &= -\frac{\lambda}{4} \int d^d x_{\perp} \int_n d^2 x_{\parallel} (N^2 + 2N) [G_n(x, x)^2], \end{aligned} \quad (24)$$

where we had used $G_n^{ij}(x, x') = \delta^{ij} G_n(x, x')$. Then we get

$$\begin{aligned} S_{\text{ent.}}(\mu^2, \lambda) &= \frac{1}{\epsilon} \frac{\lambda}{4} (N^2 + 2N) \int d^{d_\perp} x_\perp \left[\int_{1+\epsilon} d^2 x_\parallel G_{1+\epsilon}(x, x)^2 - (1 + \epsilon) \int d^2 x_\parallel G_1(x, x)^2 \right] \\ &= -\frac{N}{12} \lambda (N^2 + 2N) G_1(0) A_\perp \int \frac{d^{d_\perp} p_\perp}{(2\pi)^{d_\perp}} \frac{1}{\mu^2 + p_\perp^2}, \end{aligned} \quad (25)$$

in which we have used (16)~(18), and as a result $\int_n G_1^2$ is cancelled out by $n \int G_1^2$, leaving $\int_n G_1(0) f_n(r)$ as the leading contribution $O(\epsilon)$ in ϵ .

The following counter term is introduced to cancel the above $O(\lambda)$ corrections:

$$-\frac{1}{2} \delta \mu^2 \sum_{i=1}^N (\phi^i)^2, \quad (26)$$

where

$$\delta \mu^2 = -(N + 2) \lambda G_1(0). \quad (27)$$

This implies the renormalized mass μ is related to the bare mass μ_b by $\mu^2 = \mu_b^2 + (N + 2) \lambda G_1(0)$. This result is consistent with [23]. The derivation of (27) is summarized in Appendix for the interested readers. As a result, the entanglement entropy of the σ -model in $O(N)$ symmetric is given by (20).

In 3+1 dimensions, $d_\perp = 2$, (20) gives rise to the following divergence structure:

$$S_{\text{ent}}(\mu^2, \lambda) = \frac{A_\perp^{(2)} \Lambda^2}{48\pi} N \left\{ \ln 4 + \left(\frac{\mu^2}{\Lambda^2} \right) \ln \left(\frac{\mu^2}{\Lambda^2} \right) + O \left(\tilde{\lambda}^2, \frac{\mu^2}{\Lambda^2} \right) \right\}, \quad (\text{for } d + 1 = 4) \quad (28)$$

where $A_\perp^{(2)}$ is the area of a 2-dimensional boundary surface of A . All the correction at $O(\lambda)$ are cancelled by the counter terms.

In 2+1 dimensions, $d_\perp = 1$, and the divergence behavior arising from (20) reads

$$S_{\text{ent}}(\mu^2, \lambda) = \frac{1}{24\pi} A_\perp^{(1)} \Lambda N \left[(\log 4 + \pi) - 2\pi \left(\frac{\mu}{\Lambda} \right) + O \left(\tilde{\lambda}^2, \frac{\mu^2}{\Lambda^2} \right) \right]. \quad (\text{for } d + 1 = 3) \quad (29)$$

Here the boundary surface of A is 1-dimensional with area $A_\perp^{(1)}$.

IV. $O(N)$ σ -MODEL IN THE SYMMETRY BROKEN PHASE

The spontaneous symmetry breaking of $O(N)$ occurs when the mass squared of the scalar fields ϕ^i is tuned to $\mu^2 < 0$. Let's suppose the SSB occurs in the ϕ^N direction, i.e. ϕ^N develops

a VEV v . Then the system is left with $N - 1$ massless Goldstone bosons π^1, \dots, π^{N-1} and 1 massive scalar σ ,

$$(\phi^1, \phi^2, \dots, \phi^{N-1}, \phi^N) = (0, 0, \dots, 0, v) + (\pi^1, \pi^2, \dots, \pi^{N-1}, \sigma), \quad (30)$$

where the condensate v takes the value

$$v = \langle \phi^N \rangle = \frac{m_\sigma}{\sqrt{2}g}, \quad (31)$$

with $m_\sigma = \sqrt{-2\mu^2}$ and the new coupling constant $g = \sqrt{\lambda}$. The Euclidean Lagrangian in the SSB phase becomes

$$\begin{aligned} \mathcal{L}_E = & \sum_{i=1}^{N-1} \frac{1}{2} (\partial \pi^i)^2 + \frac{1}{2} (\partial \sigma)^2 + \frac{1}{2} m_\sigma^2 \sigma^2 \\ & + \frac{g}{\sqrt{2}} m_\sigma \left(\sum_{i=1}^{N-1} (\pi^i)^2 \sigma + \sigma^3 \right) + \frac{g^2}{4} \left(\left[\sum_{i=1}^{N-1} (\pi^i)^2 \right]^2 + \sigma^4 + 2 \sum_{i=1}^{N-1} (\pi^i)^2 \sigma^2 \right) \end{aligned} \quad (32)$$

Compared to the original $O(N)$ σ -model, the SSB phase contains not only the quartic interactions but also the cubic ones with coupling $gm_\sigma/\sqrt{2}$.

If we use the renormalized g and m_σ in the action, the partition function up to $O(g^2)$ corrections with respect to the new coupling constant g becomes,

$$\begin{aligned} \ln Z_n^{\text{SSB}} = & \ln Z_{n,0}^{\text{SSB}} - \frac{g^2}{4} \int d^{d_\perp} x_\perp \int_n d^2 x_\parallel \left\{ \left\langle \left[\sum_{i=1}^{N-1} (\pi^i)^2 \right]^2 \right\rangle + \langle \sigma^4 \rangle + 2 \sum_{i=1}^{N-1} \langle (\pi^i)^2 \sigma^2 \rangle \right\} \\ & + \frac{g^2}{4} m_\sigma^2 \int d^{d_\perp} x_\perp d^{d_\perp} x'_\perp \int_n d^2 x_\parallel d^2 x'_\parallel \left\{ \langle \sigma^3(x) \sigma^3(x') \rangle \right. \\ & \quad \left. + \sum_{i,j=1}^{N-1} \langle [\pi^i(x)]^2 \sigma(x) [\pi^j(x')]^2 \sigma(x') \rangle + 2 \sum_{i=1}^{N-1} \langle [\pi^i(x)]^2 \sigma(x) \sigma^3(x') \rangle \right\} \\ & + \text{integral of counter terms} \\ = & \ln Z_{n,0}^{\text{SSB}} \\ & - \frac{g^2}{4} \int_n d^{d+1} x \left[(N^2 - 1) G_n^\pi(x, x)^2 + 3 G_n^\sigma(x, x)^2 + 2(N - 1) G_n^\pi(x, x) G_n^\sigma(x, x) \right] \\ & + \frac{g^2}{4} m_\sigma^2 \int d^{d_\perp} x_\perp d^{d_\perp} x'_\perp \int_n d^2 x_\parallel d^2 x'_\parallel \left[6 G_n^\sigma(x', x)^3 + 2(N - 1) G_n^\pi(x', x)^2 G_n^\sigma(x', x) \right] \\ & + \text{integral of counter terms.} \end{aligned} \quad (33)$$

This is up to $O(g^2) \sim O(\lambda)$. $\ln Z_1^{\text{SSB}}$ is also given accordingly. Note that the expectation value here is taken with respect to the new vacuum in the symmetry broken phase. In terms of Feymann diagrams, the second and the third lines of (33) are depicted by the one-vertex

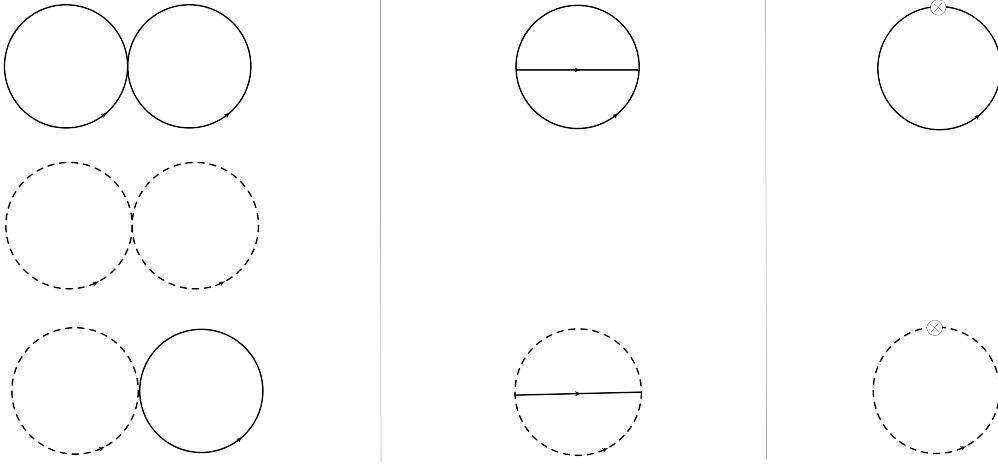


Figure 2: Feynman diagrams contributing to the two-loop corrections to the entanglement entropy in (33). The solid lines denote the massive mode σ while the dashed lines represent the Goldstone bosons π^i . The left column depicts the quartic interaction between σ 's and π^i 's. The middle column is the two-loops arising from the cubic interactions, while the right one represent the counter terms.

and two-vertices two-loops in Fig. 2 respectively. The counter terms to be added to the action to cancel the two-loop contributions are given by

$$-\frac{g^2}{2} \left(\delta^{(1)} \mu^2 + \frac{m_\sigma^2}{2} \delta^{(2)} \lambda \right) \sum_{i=1}^{N-1} (\pi^{i2})^2 - \frac{g^2}{2} \left(\delta^{(1)} \mu^2 + \frac{3m_\sigma^2}{2} \delta^{(2)} \lambda \right) \sigma^2 - \frac{gm_\sigma}{\sqrt{2}} \left(\delta^{(1)} \mu^2 + \frac{m_\sigma^2}{2} \delta^{(2)} \lambda \right) \sigma, \quad (34)$$

where $\delta^{(1)} \mu^2$, $\delta^{(2)} \lambda$ denote the coefficients of mass and coupling constant renormalization counter terms respectively, see (55) in the Appendix. Note that no wave function renormalization counter term is involved up to $O(\lambda)$ here.

In (33), we omit the cubic interactions $\langle \pi^i(x) \pi^i(x) \sigma(x) \rangle$ and $\langle \sigma(x) \sigma(x) \sigma(x) \rangle$ at $O(g)$ level, because they both vanish. Moreover, the tadpole diagrams

$$\begin{aligned} & \frac{g^2 m_\sigma^2}{4} \int_n d^{d+1} x \int_n d^{d+1} x' \{ 9 G_n^\sigma(x', x') G_n^\sigma(x', x) G_n^\sigma(x, x) \\ & + (N-1)^2 G_n^\pi(x', x') G_n^\sigma(x', x) G_n^\pi(x, x) + 6(N-1) G_n^\pi(x', x') G_n^\sigma(x', x) G_n^\sigma(x, x) \} \end{aligned} \quad (35)$$

are also dropped out, due to the requirement of the vanishing one-point function $\langle \sigma \rangle = 0$ in the vacuum of broken phase such that the one-loop corrections (i.e. the tadpoles) to the one-point function $\langle \sigma \rangle$ should be cancelled out by the counter terms. This gets rid of the

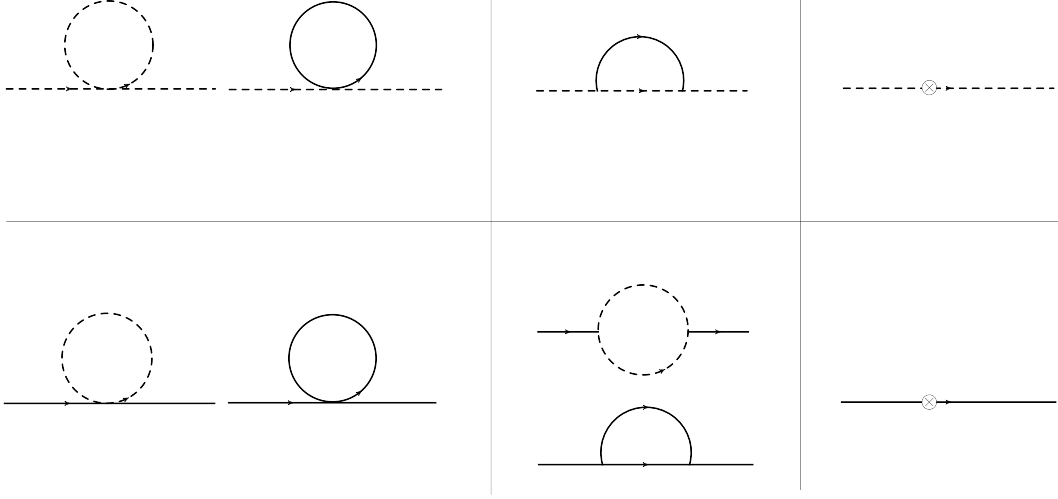


Figure 3: Feynman diagrams for the mass renormalization of π^i 's (the upper row) and σ (the lower row). The left column represents the contribution from the quartic interactions, while the middle one from the cubic interactions. The right column stands for the counter terms. The solid lines denote σ while the dashed lines π^i 's.

two-loop tadpole contribution in (35), and also gives rise to a condition

$$\delta^{(1)}\mu^2 + \frac{m_\sigma^2}{2}\delta^{(2)}\lambda = -3G_\sigma(0) - (N-1)G_\pi(0). \quad (36)$$

The Feynmann diagrams of the two-loop contributions to the renormalization are displayed in Fig. 3. The explicit expressions of the counter terms are given by

$$\begin{aligned} \delta^{(2)}\lambda &= 9L_\sigma(p^2 = m_\sigma^2) + (N-1)L_\pi(p^2 = m_\sigma^2) \\ \delta^{(1)}\mu^2 &= -3G_\sigma(0) - (N-1)G_\pi(0) - \frac{9m_\sigma^2}{2}L_\sigma(p^2 = m_\sigma^2) - \frac{m_\sigma^2}{2}(N-1)L_\pi(p^2 = m_\sigma^2) \end{aligned} \quad (37)$$

where $L_{\pi,\sigma}(p^2)$ is defined as

$$L_{\pi,\sigma}(p^2) = \int \frac{d^{d+1}q}{(2\pi)^{d+1}} G_{\pi,\sigma}(p-q) G_{\pi,\sigma}(q) \quad (38)$$

We can now use (10) to calculate the entanglement entropy. The free field contribution is

$$S_{\text{ent.}}^{\text{SSB(free)}}(m_\sigma^2) = -\frac{N-1}{12}A_\perp \int \frac{d^{d_\perp}p_\perp}{(2\pi)^{d_\perp}} \ln \frac{p_\perp^2}{\Lambda^2 + p_\perp^2} - \frac{1}{12}A_\perp \int \frac{d^{d_\perp}p_\perp}{(2\pi)^{d_\perp}} \ln \frac{m_\sigma^2 + p_\perp^2}{\Lambda^2 + p_\perp^2}. \quad (39)$$

The first term on the RHS is due to the (massless) Goldstone bosons π^1, \dots, π^{N-1} . The second term is from the massive scalar σ .

The two-loop corrections to the entanglement entropy are obtained as follows. The one-vertex sector (i.e. the second line in (33)) is computed analogously to (25), and the result is

$$\Delta S_{\text{ent}(1\text{-vt})}^{\text{SSB}}(m_\sigma^2, g^2) = -A_\perp \frac{g^2}{12} \left\{ \left(3G_1^\sigma(0) + (N-1)G_1^\pi(0) \right) \int \frac{d^{d_\perp} p_\perp}{(2\pi)^{d_\perp}} \frac{1}{m_\sigma^2 + p_\perp^2} \right. \\ \left. + \left((N-1)G_1^\sigma(0) + (N^2-1)G_1^\pi(0) \right) \int \frac{d^{d_\perp} p_\perp}{(2\pi)^{d_\perp}} \frac{1}{p_\perp^2} \right\}. \quad (40)$$

The two-vertices sector (i.e. the third line in (33)) is calculated by making use of

$$G_n(x, x') = G_1(|x - x'|) + f_n(x, x'), \quad (41)$$

where G_1 represents the $O(\epsilon^0)$ part while f_n the $O(\epsilon)$ (and the higher order) effects of the Green's function, in analogy to (16). First, one notices that $\int_n d^{d+1}x \int_n d^{d+1}x' G_1(|x - x'|)^3 - n \int_1 d^{d+1}x \int_1 d^{d+1}x' G_1(|x - x'|)^3$ is subleading at UV[23], where $G_1(|x - x'|)$ could either denotes the Green's function for σ or π^i . The argument in [23] is elaborated as follows. For convenience we change the coordinates from (x, x') to (x, y) with $y = x' - x$, such that the aforementioned subtraction becomes

$$\int_n d^{d+1}x \int_n d^{d+1}y G_1(|y|)^3 - n \int_1 d^{d+1}x \int_1 d^{d+1}y G_1(|y|)^3. \quad (42)$$

Note that the divergence of $G_1(|y|)$ occurs at small $y \sim \Lambda^{-1}$, i.e. as x' approaches x , where the fields at x' couldn't "sense" the existence of the conical point when x is not close to the conical singularity. This means that for the divergent part of $G_1(|y|)$, the integration over $\int_n d^{d+1}y$ actually takes 2π angle for y to encircle x once instead of taking $2n\pi$, and hence contributes no n factor. The only n factor in the first term of (42) is from $\int_n d^{d+1}x$. Then it is straightforward to see that, at UV, (42) schematically behaves as

$$nV\Lambda^{d-1} - nV\Lambda^{d-1},$$

and the two terms cancel out.

However, such argument breaks down when x is very close to the conical singularity, i.e. $|x| < \Lambda^{-1}$, such that the conical singularity is located within $y < \Lambda^{-1}$. In this case, both $\int_n d^{d+1}x$ and $\int_n d^{d+1}y$ give rise to an n factor, but the former only produce a $A_\perp 2n\pi/\Lambda^2$ coefficient due to restricting $|x| < \Lambda^{-1}$. So in this scenario, (42) has non-vanishing but finite contribution proportional to

$$A_\perp \frac{n^2}{\Lambda^2} \Lambda^{d-1} - A_\perp \frac{n}{\Lambda^2} \Lambda^{d-1} = A_\perp \Lambda^{d-3} (n^2 - n),$$

whose contribution is of higher order in the entanglement entropy.

In terms of (x, y) coordinates, the leading divergent contribution to $\Delta S_{\text{ent}(2\text{-vt})}^{\text{SSB}}(g^2)$ comes from

$$\begin{aligned} \frac{g^2 m_\sigma^2}{4} \int_n d^{d+1}x \int_n d^{d+1}y \left\{ 18 G_1^\sigma(|y|)^2 f_n^\sigma(x, y) \right. \\ \left. + 2(N-1) \left[2 G_1^\sigma(|y|) G_1^\pi(|y|) f_n^\pi(x, y) + G_1^\pi(|y|)^2 f_n^\sigma(x, y) \right] \right\}. \end{aligned} \quad (43)$$

By further decomposing y into y_\perp and y_\parallel , the above expression gives rise to the following corrections to the entanglement entropy:

$$\begin{aligned} \Delta S_{\text{ent}(2\text{-vt})}^{\text{SSB}}(m_\sigma^2, g^2) \sim \frac{g^2 A_\perp}{12} \int \frac{d^{d_\perp} p_\perp}{(2\pi)^{d_\perp}} \frac{m_\sigma^2}{m_\sigma^2 + p_\perp^2} \\ \times \left[9 \Gamma_\sigma(p_\perp^2) + (N-1) \Gamma_\pi(p_\perp^2) + 2(N-1) \Gamma_{\pi\sigma}(p_\perp^2) \right] \end{aligned} \quad (44)$$

where

$$\begin{aligned} \Gamma_{\pi\sigma}(p_\perp^2) &= \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k^2 + m_\sigma^2} \frac{1}{k_\parallel^2 + (\mathbf{k}_\perp + \mathbf{p}_\perp)^2}, \\ \Gamma_\pi(p_\perp^2) &= \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k^2} \frac{1}{k_\parallel^2 + (\mathbf{k}_\perp + \mathbf{p}_\perp)^2}, \\ \Gamma_\sigma(p_\perp^2) &= \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k^2 + m_\sigma^2} \frac{1}{k_\parallel^2 + (\mathbf{k}_\perp + \mathbf{p}_\perp)^2 + m_\sigma^2}. \end{aligned} \quad (45)$$

These three expressions correspond to the three Feynmann diagrams from top to bottom in the middle column of Fig. 3 respectively.

One expects that two-loop contributions from (40) and (44) will be cancelled by the counter terms in (34). This is indeed the case for the one-vertex sector (40); however there are residual terms from (44) after the cancellation,

$$\begin{aligned} \Delta S_{\text{ent}(\text{res})}^{\text{SSB}}(m_\sigma^2, g^2) \sim g^2 \frac{A_\perp}{12} \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{m_\sigma^2}{p_\perp^2 + m_\sigma^2} \left[9 D_\sigma(p_\perp^2, q^2) + (N-1) D_\pi(p_\perp^2, q^2) \right] \Big|_{q^2=m_\sigma^2} \\ + g^2 \frac{(N-1) A_\perp}{12} \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{m_\sigma^2}{p_\perp^2 + m_\sigma^2} \left[2 D_{\pi\sigma}(p_\perp^2, q^2) \right] \Big|_{q^2=0} \end{aligned} \quad (46)$$

where

$$\begin{aligned} D_\sigma(p_\perp^2, q^2) &= \Gamma_\sigma(p_\perp^2) - L_\sigma(q^2) \neq 0, \\ D_\pi(p_\perp^2, q^2) &= \Gamma_\pi(p_\perp^2) - L_\pi(q^2) \neq 0, \\ D_{\pi\sigma}(p_\perp^2, q^2) &= \Gamma_{\pi\sigma}(p_\perp^2) - L_{\pi\sigma}(q^2) \neq 0, \end{aligned} \quad (47)$$

with L_σ, L_π and $L_{\pi\sigma}$ given by (38).

The existence of these cancellation remnants means that the expansion of $S_{\text{ent.}}^{\text{SSB}}$ involves non-trivial subleading terms. In 3+1 dimensions, the entanglement entropy reads

$$S_{\text{ent.}}^{\text{SSB}}(m_\sigma^2, \lambda) = \frac{1}{48\pi} N A_\perp^{(2)} \Lambda^2 \left[\ln 4 - \frac{\frac{3}{2}N + 3}{(4\pi N)^2} \tilde{\lambda} \left(\frac{m_\sigma^2}{\Lambda^2} \right) \left[\ln \left(\frac{m_\sigma^2}{\Lambda^2} \right) \right]^2 \right. \\ \left. + \left(\frac{1}{N} - \frac{2(N-1) + 9\sqrt{5} \ln((3+\sqrt{5})/2)}{(4\pi N)^2} \tilde{\lambda} \right) \left(\frac{m_\sigma^2}{\Lambda^2} \right) \ln \left(\frac{m_\sigma^2}{\Lambda^2} \right) + O \left(\tilde{\lambda}^2, \frac{m_\sigma^2}{\Lambda^2} \right) \right] \quad (48)$$

The λ -independent part represents the exact cancellation between the quantum corrections and the counter term of the quartic sector, leaving only the tree level effects, as in the $O(N)$ symmetric phase. In this part, the leading divergence remains the same compared to the symmetric phase, contributed from $N-1$ π 's and one σ , while the N -independent log divergence arises solely from σ , since it is the only massive component in the broken phase. The λ -dependent part in (48) represents the leftover of the cancellation between the two-vertex two-loop sector (due to the cubic interactions) and the counter terms, giving rise to the log squared divergence in the subleading part, which is more divergent than the subleading log divergence in (28) in the $O(N)$ symmetric phase.

Our highest subleading divergence in (48) is log squared. This is different from the result of cubic interaction in [23]. In [23], the author argues that the leading divergence of (43) is contributed by $y \rightarrow 0$, i.e. both $y_\perp \rightarrow 0$ and $y_\parallel \rightarrow 0$, such that $f_n(x, y)$ becomes (17), and eventually (43) gives rise to log divergence as the subleading behavior of entanglement entropy. But our calculation shows that $y \neq 0$ part in $f_n(x, y)$ actually yields log squared, more divergent than log, by setting $y_\parallel \rightarrow 0$ (which allows $f_n(x, y)$ to be approximated by (17)) while preserving the $y_\perp \neq 0$ contribution and carrying out Fourier transformation. The detailed calculation for obtaining (48) from (43) is presented in the Appendix.

In 2+1 dimensions, the entanglement entropy is expressed by

$$S_{\text{ent.}}^{\text{SSB}}(m_\sigma^2, \lambda) = \frac{A_\perp^{(1)} \Lambda}{24\pi} N \left\{ (\log 4 + \pi) - \frac{1.645 + 1.805(N-1)}{8\pi N^2} \tilde{\lambda} - \frac{2\pi}{N} \left(\frac{m_\sigma}{\Lambda} \right) + O \left(\tilde{\lambda}^2, \frac{m_\sigma^2}{\Lambda^2} \right) \right\}. \quad (49)$$

where $\tilde{\lambda} = \lambda/N$. The interpretation is similar to the 3+1 dimensional case. One may notice that the subleading term in the $O(N)$ symmetric phase is cutoff independent. In the symmetry broken phase, a new term in $O(\lambda/\Lambda)$ order emerges in the subleading part compared to (29), but it doesn't change the divergence structure. Here we only supply the numerical value for the coefficient of λ/Λ . The analytic expression is given in the Appendix.

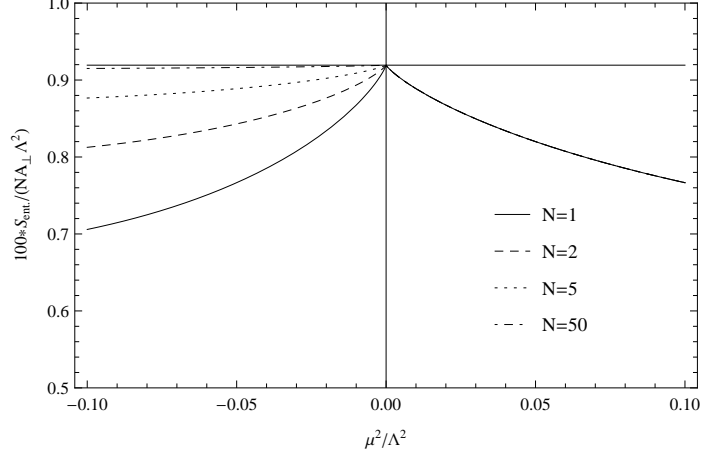


Figure 4: Entanglement entropy in units of $NA_{\perp}\Lambda^2$ magnified by a factor of 100 in the $O(N)$ σ -model with spontaneous symmetry breaking against the mass squared of fields normalized by Λ^2 in 3+1 dimensions. Note that $m_{\sigma}^2 = -2\mu^2$ in the symmetry broken phase. In this plot, we take $\tilde{\lambda} = 10^{-6}$ in order for the perturbation calculation to be valid. There is a finite maximum at the quantum phase transition point $\mu^2/\Lambda^2 = 0$.

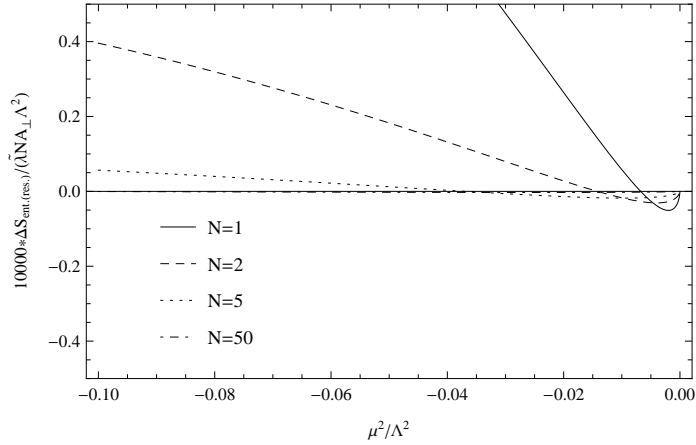


Figure 5: λ -dependent subleading part of the entanglement entropy at $O(\lambda)$ in the symmetry broken phase. Note that this plot is $\tilde{\lambda}$ -independent, as it is in units of $N\tilde{\lambda}A_{\perp}\Lambda^2$, magnified by a factor of 10000.

V. NUMERICAL RESULTS OF ENTANGLEMENT ENTROPY

In this section, we present how the entanglement entropy varies with mass and how it behaves upon quantum phase transition in the $O(N)$ σ -model, up to $O(\lambda)$, in 3+1 and 2+1 dimensions.. This result is obtained by the numerical computation.

In 3+1 dimensions, the numerical value of the entanglement entropy normalized by $A_\perp \Lambda^2 N$ against μ^2/Λ^2 in both phases are plotted in Fig. 4. The $\mu^2 > 0$ and $\mu^2 < 0$ regions are the $O(N)$ symmetric phase and the broken phase, respectively. This plot is shown in terms of renormalized μ^2 , with $\tilde{\lambda}$ set to 10^{-6} . In the $O(N)$ symmetric phase, the leading and subleading parts of S_{ent} solely arise from the free fields, and is λ -independent. On the other hand, in the symmetry broken phase, $S_{\text{ent}}^{\text{SSB}}$ contains the free field contribution and the λ -dependent remnant from counter terms cancellation of two-loop corrections due to the cubic interactions. The latter is shown in Fig. 5.

At the quantum critical point at $\mu^2 = 0$, one expects that the system acquires scaling symmetry, which gives rise to universal properties, including the scaling law of the order parameters. For the entanglement entropy up to the highest subleading term, we found a novel scaling behavior near the transition point:

$$S_{\text{ent}} \sim \begin{cases} 1 + \frac{1}{\Lambda^2 \ln 4} \mu^2 \ln \mu^2 = (\mu^2)^{\frac{\mu^2}{\Lambda^2 \ln 4}} & \text{(symmetric phase)} \\ 1 - \alpha m_\sigma^2 (\ln m_\sigma^2)^2 = (m_\sigma^2)^{-\alpha m_\sigma^2 \ln m_\sigma^2} & \text{(broken phase),} \end{cases} \quad (50)$$

where α is a $\tilde{\lambda}$ -dependent constant, $\alpha = \frac{\frac{3}{2}N+3}{(4\pi N\Lambda)^2 \ln 4} \tilde{\lambda}$. Note that $m_\sigma^2 = -2\mu^2$ for $\mu^2 < 0$. One immediately finds that the “critical exponents” are not constants as the conventional quantum critical phenomena suggests; instead they are related to the mass squared scale itself, and tends to 0 at $\mu^2 \rightarrow 0$.

Moreover, one can find in Fig. 4 that the entanglement entropy reduces as μ^2 is tuned up. This is because when the system departs from quantum critical point as μ^2 increases from 0, the correlation length $\xi \sim \mu^{-1}$ decreases, and hence the level of entanglement reduces. The entanglement entropy has a finite local maximum with a cusp at the phase transition point $\mu = 0$.

Such behavior of the entanglement entropy can also be interpreted from the point of view of lattice models. The spatial derivative term in action of field theory is regarded as the key for producing non-trivial entanglement. In the lattice models, the spatial derivative term corresponds to the difference between the fields at one site and its nearest neighbor, which is called the lattice link. Without the lattice links, the vacuum of the total system would be just the direct product of local oscillator’s vacuum at each site,

$$|\Omega\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \dots, \quad (51)$$

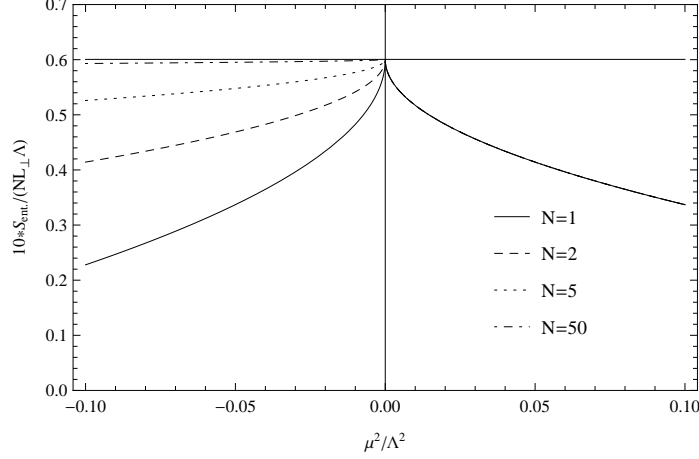


Figure 6: Entanglement entropy in units of $N L_{\perp} \Lambda^2$ magnified by a factor of 10 in the $O(N)$ σ -model with spontaneous symmetry breaking v.s. the mass squared of fields normalized by Λ^2 in 2+1 dimensions. Here $A_{\perp}^{(1)}$ is denoted by L_{\perp} . This plot takes $\tilde{\lambda} = 10^{-6}$. The entanglement entropy has a finite local maximum with a cusp at $\mu^2 = 0$.

i.e. there is no entanglement. However, when the lattice link is present, vacuum can be the non-trivial superposition of the oscillator's state at each site.

$$|\Omega\rangle = \sum_{i_1 i_2 \dots} c_{i_1 i_2 \dots} |\phi_{i_1}(1)\rangle \otimes |\phi_{i_2}(2)\rangle \otimes \dots, \quad (52)$$

where $|\phi_{i_1}(1)\rangle$ is the i -th state of oscillator at site i .

On the other hand, the mass term in the action corresponds to the harmonic potential for the oscillators at each site. As the mass increases, the potential wall becomes more steep, which enhances on-site localization and suppresses hopping. The tunneling of quantum fluctuations is also suppressed. All these effects reduce the level of entanglement against increasing μ^2 .

When the number of ϕ species $N > 1$, the Goldstones bosons emerge in the symmetry broken phase. The contribution to the entanglement entropy from each Goldstone mode is fixed and independent of μ^2 , as been demonstrated in (39). On the contrary, the massive mode always has $N = 1$ contribution. Therefore as the mass-varying contribution from σ is suppressed by increasing N , and becomes flatter. While $N \rightarrow \infty$, entanglement entropy in the broken phase is dominated by the massless Goldstone bosons, and becomes completely flat in the plot.

The numerical plot of the entanglement entropy in 2 + 1 dimensions is given in Fig. 6.

The behavior of $S_{\text{ent.}}$ and $S_{\text{ent.}}^{\text{SSB}}$ against μ^2 are very similar to the 3+1 dimensional case. The local maximum is finite with a cusp. Near the quantum phase transition point, from (29) and (29) the scaling behavior of the entanglement entropy can be derived

$$\Delta S_{\text{ent}} \sim \begin{cases} -(\mu^2)^{\frac{1}{2}} & \text{(symmetric phase)} \\ -(m_\sigma^2)^{\frac{1}{2}} & \text{(broken phase)}. \end{cases} \quad (53)$$

The critical exponent is thus a constant $\frac{1}{2}$.

VI. CONCLUSION AND OUTLOOK

In this paper, we perturbatively calculate the entanglement entropy of $O(N)$ σ -model with spontaneous symmetry breaking by tuning μ^2 , up to $O(\lambda)$ order, in 3+1 and 2+1 dimensions. We employ the renormalized mass and coupling constant in the tree level, such that the final result is expressed in terms of these renormalized parameters. The quantum corrections are cancelled by the counter terms we introduced, except the two-loop perturbations due to the cubic interactions in the symmetry broken phase. We find that in both phases, this model gives rise to the same expected leading divergence structures. The area law is preserved upon quantum phase transition. However, due to the emergence of the cubic interactions, the subleading structure in 3+1 dimensions changes from log divergence in $O(N)$ symmetric phase to log squared divergence in the broken phase. In 2+1 dimensions, the emergence of the cubic interactions also introduces a new term in the subleading component of $S_{\text{ent.}}^{\text{SSB}}$, but its degree of divergence is maintained.

We also numerically display the behavior of the entanglement entropy against μ^2 . There occurs a cusped peak at the phase boundary, marking the phase transition at $\mu^2 = 0$. While $|\mu^2|$ is tuned up, the level of entanglement reduces in both phases. This offers additional information on the quantum phase transition of $O(N)$ σ -model with the order parameter $\langle \sigma \rangle$.

Our results of the coefficients and the divergence structure of the leading behavior in $S_{\text{ent.}}$ in the $O(N)$ symmetric phase is consistent with those in [23] in terms of bare parameters, despite that we use a different renormalization prescription. This is expected, as the two renormalization prescriptions should be equivalent. Their cutoff dependence in the λ -independent subleading part is hidden in the renormalized mass, i.e. $m_r^2 = m_{\text{bare}}^2 + \delta m^2(\Lambda)$,

while because we use renormalized parameters since the tree level, in our corresponding part the cutoff dependence is manifest. But these two results in fact describe the same physics.

However, in the symmetry broken phase with cubic interactions, our subleading divergence is log squared, rather than log divergence in [23], due to the remnant from counter term cancellation. This discrepancy arises from considering $y \neq 0$ part in (43), which yields the log squared divergence compared to the log divergence from the $y \rightarrow 0$ part.

There are many interesting directions to be explored based on this work. Straight forward generalization includes introducing the gauge fields into the $O(N)$ σ -model, imposing a chemical potential, or external electric-magnetic fields, and then study the quantum phase transition in terms of entanglement. Such a system would be more complicated yet more realistic. Moreover, in high energy physics, pion gas with isospin chemical potential has quantum phase transition into Bose-Einstein condensate, which is also of interest to study from the perspective of entanglement entropy.

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Appendix: Renormalization of $O(N)$ model and Entanglement Entropy

In the symmetric phase, action of $O(N)$ model is splitted into renormalized part and counter term,

$$\begin{aligned} \mathcal{L} = & \sum_{i=1}^N \left[\frac{1}{2} (\partial \phi^i)^2 - \frac{1}{2} \mu^2 (\phi^i)^2 \right] - \frac{\lambda}{4} \left[\sum_{i=1}^N (\phi^i)^2 \right]^2 \\ & + \sum_{i=1}^N \left[\frac{1}{2} \delta Z (\partial \phi^i)^2 - \frac{1}{2} \delta \mu^2 (\phi^i)^2 \right] - \frac{\delta \lambda}{4} \left[\sum_{i=1}^N (\phi^i)^2 \right]^2 \end{aligned} \quad (54)$$

where

$$\begin{aligned} \delta Z &= O(\lambda^2), \\ \delta \mu^2 &= \lambda \delta^{(1)} \mu^2 + O(\lambda^2), \end{aligned} \quad (55)$$

$$\delta\lambda = \lambda^2 \delta^{(2)}\lambda + O(\lambda^3).$$

denote the wave-function, mass, and coupling constant counter terms expanded w.r.t. λ . The superscripts in $\delta^{(1)}\mu^2$ and $\delta^{(2)}\lambda$ label the order in λ expansion. It can be demonstrated that wave-function renormalization counter term δZ and coupling renormalization counter term are no less than second order. Only the mass renormalization counter term $\delta\mu^2$ is involved in $O(\lambda)$. So we have,

$$\begin{aligned} \mathcal{L} = & \sum_{i=1}^N \left[\frac{1}{2} (\partial\phi^i)^2 - \frac{1}{2}\mu^2 (\phi^i)^2 \right] - \frac{\lambda}{4} \left[\sum_{i=1}^N (\phi^i)^2 \right]^2 \\ & - \sum_{i=1}^N \left[\frac{1}{2}\delta\mu^2 (\phi^i)^2 \right]. \end{aligned} \quad (56)$$

After the spontaneous symmetry breaking, however, the Lagrangian becomes order level is

$$\begin{aligned} \mathcal{L} = & \sum_{i=1}^{N-1} \left[\frac{1}{2} (\partial\pi^i)^2 \right] + \left[\frac{1}{2} (\partial\sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 \right] \\ & - \frac{\lambda}{4} \left[\sum_{i=1}^{N-1} (\pi^i)^2 \right]^2 - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2} \left[\sum_{i=1}^{N-1} (\pi^i)^2 \right] \sigma^2 \\ & - \sqrt{\frac{\lambda}{2}}m_\sigma \left[\sum_{i=1}^{N-1} (\pi^i)^2 \right] \sigma - \sqrt{\frac{\lambda}{2}}m_\sigma\sigma^3 \\ & - \frac{\lambda}{2} \left[\delta^{(1)}\mu^2 + \frac{1}{2}m_\sigma^2\delta^{(2)}\lambda \right] \sum_{i=1}^{N-1} [(\pi^i)^2] \\ & - \frac{\lambda}{2} \left[\delta^{(1)}\mu^2 + \frac{3}{2}m_\sigma^2\delta^{(2)}\lambda \right] \sigma^2 \\ & - \sqrt{\frac{\lambda}{2}}m_\sigma \left[\delta^{(1)}\mu^2 + \frac{1}{2}m_\sigma^2\delta^{(2)}\lambda \right] \sigma + O(\lambda^2) \end{aligned} \quad (57)$$

where $m_\sigma = \sqrt{-2\mu^2}$ and the fields split into

$$(\phi^1, \phi^2, \dots, \phi^{N-1}, \phi^N) = (0, 0, \dots, 0, \frac{m_\sigma}{\sqrt{2\lambda}}) + (\pi^1, \pi^2, \dots, \pi^{N-1}, \sigma). \quad (58)$$

Here one can find that the coupling renormalization counter term at $O(\lambda^2)$ comes into play in the symmetry broken Lagrangian at $O(\lambda)$. The wave-function renormalization counter term will involve non-zero part at $O(\lambda)$ here. However, it is finite and thus irrelevant to our calculation because one can always choose a suitable coupling constant renormalization to make it vanish.

The renormalization scheme depends on the renormalization condition. We choose renormalization condition at pole mass. Assuming the propagator has the form as,

$$G^{-1}(p^2) = p^2 + M^2(p^2), \quad (59)$$

In the symmetric phase, the mass of ϕ is defined by renormalization condition,

$$M_\phi^2(p^2 = \mu^2) = \mu^2, \quad (60)$$

while in the broken phase, we fix the mass of σ using renormalization condition,

$$M_\sigma^2(p^2 = m_\sigma^2) = m_\sigma^2 = -2\mu^2. \quad (61)$$

Thus the tuning mass is always the pole mass that we can measure in the lab.

The entanglement entropy of the scalar field theory in $3+1$ dimensions is UV divergent,

$$\begin{aligned} S_{\text{ent.}}(\mu^2, \lambda) = & \#A_\perp^{(2)}\Lambda^2 [(a_0 + a_1\lambda)) \\ & + (b_0 + b_1\lambda)) \left(\frac{|\mu^2|}{\Lambda^2} \right) \ln \left(\frac{|\mu^2|}{\Lambda^2} \right) + O \left(\lambda^2, \frac{|\mu^2|}{\Lambda^2} \right)] \end{aligned} \quad (62)$$

where Λ is cut-off of momentum. This is indeed the case for the $O(N)$ σ -model in symmetric phase. As $|\mu^2|/\Lambda^2 \rightarrow 0$, the leading order is $a_0 + a_1\lambda + O(\lambda^2)$, while the subleading order is

$$(b_0 + b_1\lambda + O(\lambda^2)) \left(\frac{|\mu^2|}{\Lambda^2} \right) \ln \left(\frac{|\mu^2|}{\Lambda^2} \right).$$

Since the expression of entanglement entropy depends on the regularization, i.e. the momentum cut-off, it depends on the renormalization scheme.

In the $O(N)$ symmetric phase, the wave-function and coupling renormalization counter terms are not involved, so renormalization condition at $O(\lambda)$ level is given by

$$M_{\phi,(0)}^2 + M_{\phi,(1)}^2\lambda + O(\lambda^2) = \mu^2 \quad (63)$$

where

$$M_{\phi,(0)}^2 = \mu^2, \quad (64)$$

$$M_{\phi,(1)}^2 = N \left[\delta^{(1)}\mu^2 + (N+2)G_\phi(0) \right]. \quad (65)$$

Thus, we have

$$\delta\mu^2 = -\lambda(N+2)G_\phi(0) + O(\lambda^2). \quad (66)$$

The $S_{\text{ent.}}$ can be expressed by the transverse mass m_\perp^2 ,

$$S_{\text{ent.}}(\mu^2 > 0) = -\frac{NA_\perp^{(2)}}{12} \int \frac{d^2p_\perp}{(2\pi)^2} \ln \frac{p_\perp^2 + m_\perp^2(\mu^2)}{p_\perp^2 + \Lambda^2}. \quad (67)$$

With the counter term, the transverse mass is just renormalized mass,

$$\begin{aligned} m_{\perp}^2 &= \mu^2 + \lambda(N+2)G_{\phi}(0) + \lambda\delta^{(1)}\mu^2 + O(\lambda^2) \\ &= \mu^2 + O(\lambda^2). \end{aligned} \quad (68)$$

If we express the entanglement entropy in the following general form,

$$S_{\text{ent.}}(\mu^2 > 0) = \#A_{\perp}^{(2)}\Lambda^2 \left[(a_0 + a_1\lambda) + (b_0 + b_1\lambda) \left(\frac{\mu^2}{\Lambda^2} \right) \ln \left(\frac{\mu^2}{\Lambda^2} \right) + O \left(\lambda^2, \frac{\mu^2}{\Lambda^2} \right) \right], \quad (69)$$

the coefficients then are

$$\begin{aligned} a_0 &= N \log 4; \\ b_0 &= N; \\ a_1 &= b_1 = 0. \end{aligned} \quad (70)$$

After the $O(N)$ symmetry is spontaneously broken, we have to consider coupling constant renormalization counter term. Here we use the pole mass renormalization condition along with the requirement of one-point function cancellation $\langle \sigma(x) \rangle = V_{\sigma}G_{\sigma}(x) = 0$:

$$M_{\sigma,(0)}^2 + M_{\sigma,(1)}^2(p^2 = m_{\sigma}^2)\lambda + O(\lambda^2) = m_{\sigma}^2, \quad (71)$$

$$V_{\sigma,(0)} + V_{\sigma,(1)}\sqrt{\lambda} + O(\lambda) = 0 \quad (72)$$

where

$$M_{\sigma,(0)}^2 = m_{\sigma}^2, \quad (73)$$

$$\begin{aligned} M_{\sigma,(1)}^2(p^2 = m_{\sigma}^2) &= 3G_{\sigma}(0) + (N-1)G_{\pi}(0) \\ &\quad - 9m_{\sigma}^2 L_{\sigma}(p^2 = m_{\sigma}^2) - (N-1)m_{\sigma}^2 L_{\pi}(p^2 = m_{\sigma}^2) \\ &\quad + \left(\delta^{(1)}\mu^2 + \frac{3}{2}m_{\sigma}^2\delta^{(2)}\lambda \right), \end{aligned} \quad (74)$$

$$V_{\sigma,(0)} = 0, \quad (75)$$

$$\begin{aligned} V_{\sigma,(1)} &= -\frac{m_{\sigma}}{\sqrt{2}} \left[3G_{\sigma}(0) + (N-1)G_{\pi}(0) \right. \\ &\quad \left. + \left(\delta^{(1)}\mu^2 + \frac{1}{2}m_{\sigma}^2\delta^{(2)}\lambda \right) \right], \end{aligned} \quad (76)$$

in which $L(p^2)$ is defined by

$$L(p^2) = \int \frac{d^4q}{(2\pi)^4} G(p-q)G(q). \quad (77)$$

Thus, we have

$$\begin{aligned}\delta\mu^2 = & -\lambda [3G_\sigma(0) + (N-1)G_\pi(0) \\ & + \frac{9}{2}m_\sigma^2 L_\sigma(p^2 = m_\sigma^2) + \frac{1}{2}(N-1)m_\sigma^2 L_\pi(p^2 = m_\sigma^2)] + O(\lambda^2),\end{aligned}\quad (78)$$

$$\delta\lambda = \lambda^2 [9L_\sigma(p^2 = m_\sigma^2) + (N-1)L_\pi(p^2 = m_\sigma^2)] + O(\lambda^3). \quad (79)$$

Now the entanglement entropy reads

$$\begin{aligned}S_{\text{ent.}}(\mu^2 < 0) = & -\frac{A_\perp^{(2)}}{12} \int \frac{d^2 p_\perp}{(2\pi)^2} \ln \frac{p_\perp^2 + m_{\sigma,\perp}^2(m_\sigma^2)}{p_\perp^2 + \Lambda^2} \\ & -\frac{(N-1)A_\perp^{(2)}}{12} \int \frac{d^2 p_\perp}{(2\pi)^2} \ln \frac{p_\perp^2 + m_{\pi,\perp}^2(m_\sigma^2)}{p_\perp^2 + \Lambda^2} \\ & + \lambda \frac{A_\perp^{(2)}}{12} \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{m_\sigma^2}{p_\perp^2 + m_\sigma^2} [D_\sigma(p_\perp^2, q^2) + (N-1)D_\pi(p_\perp^2, q^2)] \Big|_{q^2=m_\sigma^2} \\ & + \lambda \frac{(N-1)A_\perp^{(2)}}{12} \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{m_\sigma^2}{p_\perp^2 + m_\sigma^2} [D_{\pi\sigma}(p_\perp^2, q^2)] \Big|_{q^2=0} + O(\lambda^2)\end{aligned}\quad (80)$$

where

$$\begin{aligned}D_\sigma(p_\perp^2, q^2) = & 9 \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k^2 + m_\sigma^2} \left[\frac{1}{k_\parallel^2 + (\mathbf{k}_\perp + \mathbf{p}_\perp)^2 + m_\sigma^2} - \frac{1}{(\mathbf{k} + \mathbf{q})^2 + m_\sigma^2} \right] \\ = & 9 [\Gamma_\sigma(p_\perp^2) - L_\sigma(q^2)];\end{aligned}\quad (81)$$

$$\begin{aligned}D_\pi(p_\perp^2, q^2) = & \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k^2} \left[\frac{1}{k_\parallel^2 + (\mathbf{k}_\perp + \mathbf{p}_\perp)^2} - \frac{1}{(\mathbf{k} + \mathbf{q})^2} \right] \\ = & \Gamma_\pi(p_\perp^2) - L_\pi(q^2);\end{aligned}\quad (82)$$

$$\begin{aligned}D_{\pi\sigma}(p_\perp^2, q^2) = & 2 \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k^2 + m_\sigma^2} \left[\frac{1}{k_\parallel^2 + (\mathbf{k}_\perp + \mathbf{p}_\perp)^2} - \frac{1}{(\mathbf{k} + \mathbf{q})^2} \right] \\ = & 2 [\Gamma_{\pi\sigma}(p_\perp^2) - L_{\pi\sigma}(q^2)].\end{aligned}\quad (83)$$

The explicit calculation of D is shows that

$$\begin{aligned}D_\sigma(p_\perp^2, q^2) = & \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{9}{k^2 + m_\sigma^2} \left[\frac{1}{k^2 + m_\sigma^2 + 2\mathbf{k}_\perp \cdot \mathbf{p}_\perp + p_\perp^2} - \frac{1}{k^2 + m_\sigma^2 + 2\mathbf{k} \cdot \mathbf{q} + q^2} \right] \\ = & 9 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \left[\left(\frac{1}{l^2 + m_\sigma^2 - x^2 p_\perp^2 + x p_\perp^2} \right)^2 - \left(\frac{1}{l^2 + m_\sigma^2 - x^2 q^2 + x q^2} \right)^2 \right] \\ = & \frac{9}{(4\pi)^2} \int_0^1 dx \ln \left(\frac{m_\sigma^2 + x(1-x)q^2}{m_\sigma^2 + x(1-x)p_\perp^2} \right); \\ D_\pi(p_\perp^2, q^2) = & \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k^2} \left[\frac{1}{k^2 + 2\mathbf{k}_\perp \cdot \mathbf{p}_\perp + p_\perp^2} - \frac{1}{k^2 + 2\mathbf{k} \cdot \mathbf{q} + q^2} \right]\end{aligned}\quad (84)$$

$$\begin{aligned}
&= \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \left[\left(\frac{1}{l^2 - x^2 p_\perp^2 + x p_\perp^2} \right)^2 - \left(\frac{1}{l^2 - x^2 q^2 + x q^2} \right)^2 \right] \\
&= \frac{1}{(4\pi)^2} \ln \left(\frac{q^2}{p_\perp^2} \right); \tag{85}
\end{aligned}$$

$$\begin{aligned}
D_{\pi\sigma}(p_\perp^2, q^2) &= 2 \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k^2 + m_\sigma^2} \left[\frac{1}{k^2 + 2\mathbf{k}_\perp \cdot \mathbf{p}_\perp + p_\perp^2} - \frac{1}{k^2 + 2\mathbf{k} \cdot \mathbf{q} + q^2} \right] \\
&= 2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \left[\left(\frac{1}{l^2 + (1-x)m_\sigma^2 - x^2 p_\perp^2 + x p_\perp^2} \right)^2 \right. \\
&\quad \left. - \left(\frac{1}{l^2 + (1-x)m_\sigma^2 - x^2 q^2 + x q^2} \right)^2 \right] \\
&= \frac{2}{(4\pi)^2} \int_0^1 dx \ln \left[\frac{m_\sigma^2 + x q^2}{m_\sigma^2 + x p_\perp^2} \right]. \tag{86}
\end{aligned}$$

The transverse masses of π and σ are

$$\begin{aligned}
m_{\sigma,\perp}^2 &= m_\sigma^2 + \lambda \left[3G_\sigma(0) + (N-1)G_\pi(0) - 9m_\sigma^2 L_\sigma(p^2 = m_\sigma^2) - (N-1)m_\sigma^2 L_\pi(p^2 = m_\sigma^2) \right] \\
&\quad + \lambda \left[\delta^{(1)} \mu^2 + \frac{3}{2} m_\sigma^2 \delta^{(2)} \lambda \right] + O(\lambda^2), \tag{87}
\end{aligned}$$

$$\begin{aligned}
m_{\pi,\perp}^2 &= \lambda \left[G_\sigma(0) + (N+1)G_\pi(0) - 2m_\sigma^2 L_{\pi\sigma}(p^2 = 0) \right] \\
&\quad + \lambda \left[\delta^{(1)} \mu^2 + \frac{1}{2} m_\sigma^2 \delta^{(2)} \lambda \right] + O(\lambda^2). \tag{88}
\end{aligned}$$

The mass of π is always zero. This gives rise to the condition

$$\delta^{(1)} \mu^2 + \frac{1}{2} m_\sigma^2 \delta^{(2)} \lambda = -3G_\sigma(0) - (N-1)G_\pi(0). \tag{89}$$

One can demonstrate that

$$M_\pi^2(p^2 = 0) = 0 + O(\lambda^2), \tag{90}$$

$$m_{\pi,\perp}^2 = 0 + O(\lambda^2). \tag{91}$$

And, by normalizing m_σ at $q^2 = m_\sigma^2$, we have

$$m_{\sigma,\perp}^2 = m_\sigma^2 + O(\lambda^2). \tag{92}$$

The explicit calculation of residual part of the entanglement entropy that cannot be absorbed by mass renormalization shows that $\Delta S_{\text{ent.}}^{\text{res.}}$ is given by

$$\Delta S_{\text{ent.}}^{\text{res.}} = +\lambda \frac{A_\perp^{(2)}}{12} \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{m_\sigma^2}{p_\perp^2 + m_\sigma^2} \left[D_\sigma(p_\perp^2, q^2) + (N-1)D_\pi(p_\perp^2, q^2) \right] \Big|_{q^2=m_\sigma^2}$$

$$\begin{aligned}
& + \lambda \frac{(N-1)A_{\perp}^{(2)}}{12} \int \frac{d^2 p_{\perp}}{(2\pi)^2} \frac{m_{\sigma}^2}{p_{\perp}^2} \left[D_{\pi\sigma}(p_{\perp}^2, q^2) \right] \Big|_{q^2=0} + O(\lambda^2) \\
& = \lambda \frac{A_{\perp}^{(2)}}{12} \int \frac{d^2 p_{\perp}}{(2\pi)^2} \frac{1}{(4\pi)^2} \left\{ \frac{m_{\sigma}^2}{p_{\perp}^2 + m_{\sigma}^2} \left[9 \int_0^1 dx \ln \left(\frac{m_{\sigma}^2 + x(1-x)m_{\sigma}^2}{m_{\sigma}^2 + x(1-x)p_{\perp}^2} \right) + (N-1) \ln \left(\frac{m_{\sigma}^2}{p_{\perp}^2} \right) \right] \right. \\
& \quad \left. + \frac{m_{\sigma}^2}{p_{\perp}^2} \left[2(N-1) \int_0^1 dx \ln \left(\frac{m_{\sigma}^2}{m_{\sigma}^2 + xp_{\perp}^2} \right) \right] \right\} + O(\lambda^2) \\
& = \frac{A_{\perp}^{(2)}\Lambda^2}{48\pi} \left\{ \frac{\lambda}{(4\pi)^2} \int_0^1 dt \left\{ \frac{\tilde{m}_{\sigma}^2}{t + \tilde{m}_{\sigma}^2} \left[9 \int_0^1 dx \ln \left(\frac{\tilde{m}_{\sigma}^2 + x(1-x)\tilde{m}_{\sigma}^2}{\tilde{m}_{\sigma}^2 + x(1-x)t} \right) + (N-1) \ln \left(\frac{\tilde{m}_{\sigma}^2}{t} \right) \right] \right. \right. \\
& \quad \left. \left. + \frac{\tilde{m}_{\sigma}^2}{t} \left[2(N-1) \int_0^1 dx \ln \left(\frac{\tilde{m}_{\sigma}^2}{\tilde{m}_{\sigma}^2 + xt} \right) \right] \right\} + O(\lambda^2) \right\} \\
& = \frac{A_{\perp}^{(2)}\Lambda^2}{48\pi} \left\{ \frac{\lambda}{(4\pi)^2} \int_0^1 dt \left\{ \frac{\tilde{m}_{\sigma}^2}{t + \tilde{m}_{\sigma}^2} \left[-18 \frac{\operatorname{arcth} \sqrt{\frac{t}{t+4\tilde{m}_{\sigma}^2}}}{\sqrt{\frac{t}{t+4\tilde{m}_{\sigma}^2}}} + 9\sqrt{5} \ln \left(\frac{3+\sqrt{5}}{2} \right) + (N-1) \ln \left(\frac{\tilde{m}_{\sigma}^2}{t} \right) \right] \right. \right. \\
& \quad \left. \left. + \frac{\tilde{m}_{\sigma}^2}{t} 2(N-1) \left[1 + \left(1 + \frac{\tilde{m}_{\sigma}^2}{t} \right) \ln \tilde{m}_{\sigma}^2 / (t + \tilde{m}_{\sigma}^2) \right] \right\} + O(\lambda^2) \right\} \\
& = \frac{A_{\perp}^{(2)}\Lambda^2}{48\pi} \left\{ \frac{\lambda}{(4\pi)^2} \left\{ -18\tilde{m}_{\sigma}^2 \operatorname{Sl}_2(\tilde{m}_{\sigma}^2) + \left[9\sqrt{5} \ln \left(\frac{3+\sqrt{5}}{2} \right) \right] \tilde{m}_{\sigma}^2 \ln \left(\frac{\tilde{m}_{\sigma}^2 + 1}{\tilde{m}_{\sigma}^2} \right) \right. \right. \\
& \quad \left. \left. + (N-1)\tilde{m}_{\sigma}^2 \left[\ln \left(\frac{\tilde{m}_{\sigma}^2 + 1}{\tilde{m}_{\sigma}^2} \right) \ln(\tilde{m}_{\sigma}^2) - \operatorname{Li}_2 \left(-\frac{1}{\tilde{m}_{\sigma}^2} \right) \right] \right. \right. \\
& \quad \left. \left. + 2(N-1)\tilde{m}_{\sigma}^2 \left[-1 + \left(1 + \tilde{m}_{\sigma}^2 \right) \ln \left(\frac{\tilde{m}_{\sigma}^2 + 1}{\tilde{m}_{\sigma}^2} \right) + \operatorname{Li}_2 \left(-\frac{1}{\tilde{m}_{\sigma}^2} \right) \right] \right\} + O(\lambda^2) \right\} \\
& = \frac{A_{\perp}^{(2)}\Lambda^2}{48\pi} \left\{ \frac{\lambda}{(4\pi)^2} \left\{ -\frac{9}{2}\tilde{m}_{\sigma}^2 \ln^2(\tilde{m}_{\sigma}^2) + O(\tilde{m}_{\sigma}^2) - \left[9\sqrt{5} \ln \left(\frac{3+\sqrt{5}}{2} \right) \right] \tilde{m}_{\sigma}^2 \ln(\tilde{m}_{\sigma}^2) + O(\tilde{m}_{\sigma}^2) \right. \right. \\
& \quad \left. \left. + (N-1)\tilde{m}_{\sigma}^2 \left[-\ln^2(\tilde{m}_{\sigma}^2) + \frac{1}{2}\ln^2(\tilde{m}_{\sigma}^2) + O(\tilde{m}_{\sigma}^2) \right] \right. \right. \\
& \quad \left. \left. + 2(N-1)\tilde{m}_{\sigma}^2 \left[-\ln(\tilde{m}_{\sigma}^2) - \frac{1}{2}\ln^2(\tilde{m}_{\sigma}^2) + O(\tilde{m}_{\sigma}^2) \right] \right\} + O(\lambda^2) \right\} \\
& = \frac{A_{\perp}^{(2)}\Lambda^2}{48\pi} \left\{ \frac{\lambda}{(4\pi)^2} \left\{ -\left[\frac{9}{2} + \frac{3}{2}(N-1) \right] \tilde{m}_{\sigma}^2 \ln^2(\tilde{m}_{\sigma}^2) \right. \right. \\
& \quad \left. \left. - \left[9\sqrt{5} \ln \left(\frac{3+\sqrt{5}}{2} \right) + 2(N-1) \right] \tilde{m}_{\sigma}^2 \ln(\tilde{m}_{\sigma}^2) \right\} + O(\lambda^2, \tilde{m}_{\sigma}^2) \right\},
\end{aligned}$$

where Li_2 is the polylog function $\operatorname{Li}_{\alpha=2}$, and we also define the function of integral for σ loop by

$$\operatorname{Sl}_2(\tilde{m}_{\sigma}^2) = \int_0^1 dt \frac{1}{t + \tilde{m}_{\sigma}^2} \left[\frac{\operatorname{arcth} \sqrt{\frac{t}{t+4\tilde{m}_{\sigma}^2}}}{\sqrt{\frac{t}{t+4\tilde{m}_{\sigma}^2}}} \right], \quad (93)$$

such that

$$\lim_{x \rightarrow 0} \frac{\operatorname{Sl}_2(x)}{\ln^2 x} = \frac{1}{4}, \quad (94)$$

$$\lim_{x \rightarrow 0} \frac{-\text{Li}_2(-\frac{1}{x})}{\ln^2 x} = \frac{1}{2}. \quad (95)$$

To summarize,

$$\begin{aligned} \Delta S_{\text{ent.}}^{\text{res.}} &= +\lambda \frac{A_{\perp}^{(2)}}{12} \int \frac{d^2 p_{\perp}}{(2\pi)^2} \frac{m_{\sigma}^2}{p_{\perp}^2 + m_{\sigma}^2} \left[D_{\sigma}(p_{\perp}^2, q^2) + (N-1) D_{\pi}(p_{\perp}^2, q^2) \right] \Big|_{q^2=m_{\sigma}^2} \\ &\quad + \lambda \frac{(N-1)A_{\perp}^{(2)}}{12} \int \frac{d^2 p_{\perp}}{(2\pi)^2} \frac{m_{\sigma}^2}{p_{\perp}^2} \left[D_{\pi\sigma}(p_{\perp}^2, q^2) \right] \Big|_{q^2=0} + O(\lambda^2) \\ &= \frac{A_{\perp}^{(2)}\Lambda^2}{48\pi} \left\{ \frac{\lambda}{(4\pi)^2} \int_0^1 dt \left\{ \frac{\tilde{m}_{\sigma}^2}{t + \tilde{m}_{\sigma}^2} \left[9 \int_0^1 dx \ln \left(\frac{\tilde{m}_{\sigma}^2 + x(1-x)\tilde{m}_{\sigma}^2}{\tilde{m}_{\sigma}^2 + x(1-x)t} \right) + (N-1) \ln \left(\frac{\tilde{m}_{\sigma}^2}{t} \right) \right] \right. \right. \\ &\quad \left. \left. + \frac{\tilde{m}_{\sigma}^2}{t} \left[2(N-1) \int_0^1 dx \ln \left(\frac{\tilde{m}_{\sigma}^2}{\tilde{m}_{\sigma}^2 + xt} \right) \right] \right\} + O(\lambda^2) \right\} \\ &= \frac{A_{\perp}^{(2)}\Lambda^2}{48\pi} \left[c'_N \lambda \left(\frac{m_{\sigma}^2}{\Lambda^2} \right) \ln^2 \left(\frac{m_{\sigma}^2}{\Lambda^2} \right) + c_N \lambda \left(\frac{m_{\sigma}^2}{\Lambda^2} \right) \ln \left(\frac{m_{\sigma}^2}{\Lambda^2} \right) + O(\lambda^2, \frac{m_{\sigma}^2}{\Lambda^2}) \right], \end{aligned} \quad (96)$$

where

$$c_N = -\frac{1}{(4\pi)^2} [\beta + 2(N-1)], \quad (97)$$

$$c'_N = -\frac{1}{(4\pi)^2} \left[\beta' + \frac{3}{2}(N-1) \right], \quad (98)$$

with

$$\beta = 9\sqrt{5} \ln \frac{3 + \sqrt{5}}{2}, \quad (99)$$

$$\beta' = 18 \lim_{x \rightarrow 0} \frac{1}{\ln^2 x} \int_0^1 dt \frac{1}{t+x} \frac{\text{arcth} \sqrt{\frac{t}{t+4x}}}{\sqrt{\frac{t}{t+4x}}} = \frac{9}{2}. \quad (100)$$

At last, we can see that there is an extra order that is more divergent than $m_{\sigma}^2 \log(m_{\sigma}^2/\Lambda^2)$ but less divergent than 1 as $m_{\sigma}^2/\Lambda^2 \rightarrow 0$. This order comes from all the two-vertex loops due to the cubic interactions.

In the $2+1$ dimensional case, one can start from second equality on the R.H.S. of (84), (85) and (86), but with $d^4 l \rightarrow d^3 l$, and obtain

$$D_{\sigma}(p_{\perp}^2, q^2) = \frac{9}{2\pi} \left[\frac{\Lambda}{p_{\perp}} \arctan \left(\frac{p_{\perp}}{2m_{\sigma}} \right) - \frac{\Lambda}{q} \arctan \left(\frac{q}{2m_{\sigma}} \right) \right]; \quad (101)$$

$$D_{\pi}(p_{\perp}^2, q^2) = \frac{1}{2\pi} \left[\frac{\Lambda}{p_{\perp}} \arctan \left(\frac{p_{\perp}}{2m_{\pi}} \right) - \frac{\Lambda}{q} \arctan \left(\frac{q}{2m_{\pi}} \right) \right]; \quad (102)$$

$$\begin{aligned} D_{\pi\sigma}(p_{\perp}^2, q^2) &= \frac{1}{2\pi} \left[\frac{\Lambda}{p_{\perp}} \left[\arctan \left(\frac{p_{\perp}^2 + m_{\sigma}^2 - m_{\pi}^2}{2p_{\perp}m_{\pi}} \right) + \arctan \left(\frac{p_{\perp}^2 + m_{\pi}^2 - m_{\sigma}^2}{2p_{\perp}m_{\sigma}} \right) \right] \right. \\ &\quad \left. - (p_{\perp} \leftrightarrow q) \right] \end{aligned} \quad (103)$$

The result is

$$\begin{aligned}
\Delta S_{\text{ent.}}^{\text{res.}} &\sim \frac{\lambda}{\Lambda} A_{\perp}^{(1)} \int dp_{\perp} \left\{ \frac{m_{\sigma}^2}{p_{\perp}^2 + m_{\sigma}^2} [D_{\sigma}(p_{\perp}^2, q^2)] \Big|_{q^2=m_{\sigma}^2} + \frac{m_{\sigma}^2}{p_{\perp}^2 + m_{\sigma}^2} [(N-1)D_{\pi}(p_{\perp}^2, q^2)] \Big|_{q^2=m_{\sigma}^2} \right. \\
&\quad \left. + (N-1) \frac{m_{\sigma}^2}{p_{\perp}^2} [D_{\pi\sigma}(p_{\perp}^2, q^2)] \Big|_{q^2=0} \right\} + O(\lambda^2) \\
&\sim \lambda A_{\perp}^{(1)} \frac{m_{\sigma}^2}{8\pi} \int dp_{\perp} \left\{ \frac{36}{p_{\perp}^2 + m_{\sigma}^2} \left[\frac{1}{p_{\perp}} \arctan \left(\frac{p_{\perp}}{2m_{\sigma}} \right) - \frac{1}{m_{\sigma}} \arctan \left(\frac{1}{2} \right) \right] \right. \\
&\quad + \frac{4(N-1)}{p_{\perp}^2 + m_{\sigma}^2} \left[\frac{1}{p_{\perp}} \arctan \left(\frac{p_{\perp}}{2m_{\pi}} \right) - \frac{1}{m_{\sigma}} \arctan \left(\frac{m_{\sigma}}{2m_{\pi}} \right) \right] \\
&\quad + \frac{4(N-1)}{p_{\perp}^2 + m_{\pi}^2} \left[\left[\frac{1}{p_{\perp}} \arctan \left(\frac{p_{\perp}^2 + m_{\sigma}^2 - m_{\pi}^2}{2p_{\perp}m_{\pi}} \right) + \frac{1}{p_{\perp}} \arctan \left(\frac{p_{\perp}^2 + m_{\pi}^2 - m_{\sigma}^2}{2p_{\perp}m_{\sigma}} \right) \right] \right. \\
&\quad \left. \left. - \frac{2}{m_{\sigma} + m_{\pi}} \right] \right\} + O(\lambda^2) \\
&\sim \lambda A_{\perp}^{(1)} \frac{1}{8\pi} \left[\tilde{m}_{\sigma}^2 36 \text{Sl}(\tilde{m}_{\sigma}^2) + 4(N-1) (\tilde{m}_{\sigma}^2 \text{Pl}(\tilde{m}_{\sigma}^2) + \tilde{m}_{\sigma}^2 \text{Ml}(\tilde{m}_{\sigma}^2)) \right] + O(\lambda^2), \tag{104}
\end{aligned}$$

where

$$\text{Sl}(\tilde{m}_{\sigma}^2) = \int_0^1 dt \left\{ \frac{1}{t^2 + \tilde{m}_{\sigma}^2} \left[\frac{1}{t} \arctan \left[\frac{1}{2} \left(\frac{t}{\tilde{m}_{\sigma}} \right) \right] - \frac{1}{\tilde{m}_{\sigma}} \arctan \left(\frac{1}{2} \right) \right] \right\}, \tag{105}$$

$$\text{Pl}(\tilde{m}_{\sigma}^2) = \lim_{\tilde{m}_{\pi} \rightarrow 0} \int_0^1 dt \frac{1}{t^2 + \tilde{m}_{\sigma}^2} \left[\frac{1}{t} \arctan \left[\frac{1}{2} \left(\frac{t}{\tilde{m}_{\pi}} \right) \right] - \frac{1}{\tilde{m}_{\sigma}} \arctan \left[\frac{1}{2} \left(\frac{\tilde{m}_{\sigma}}{\tilde{m}_{\pi}} \right) \right] \right], \tag{106}$$

$$\begin{aligned}
\text{Ml}(\tilde{m}_{\sigma}^2) &= \lim_{\tilde{m}_{\pi} \rightarrow 0} \int_0^1 dt \frac{1}{t^2 + \tilde{m}_{\pi}^2} \left\{ \left[\frac{1}{t} \arctan \left(\frac{t^2 + \tilde{m}_{\sigma}^2 - \tilde{m}_{\pi}^2}{2t\tilde{m}_{\pi}} \right) + \frac{1}{t} \arctan \left(\frac{t^2 + \tilde{m}_{\pi}^2 - \tilde{m}_{\sigma}^2}{2t\tilde{m}_{\sigma}} \right) \right] \right. \\
&\quad \left. - \frac{2}{\tilde{m}_{\sigma} + \tilde{m}_{\pi}} \right\}. \tag{107}
\end{aligned}$$

To summarize, the entanglement entropy in the symmetry broken phase in 2+1 dimensions is

$$\begin{aligned}
S_{\text{ent.}}(\mu^2 < 0) &= \frac{1}{24\pi} N A_{\perp}^{(1)} \Lambda \left\{ \left[(\log 4 + \pi) - \frac{2\pi}{N} \left(\frac{m_{\sigma}}{\Lambda} \right) \right] \right. \\
&\quad \left. + c_3 \frac{\lambda}{\Lambda} + O \left(\lambda^2, \frac{m_{\sigma}^2}{\Lambda^2} \right) \right\}, \tag{108}
\end{aligned}$$

with

$$c_3 = -\frac{1}{8\pi N} [\beta_3 + \beta_3'(N-1)] \tag{109}$$

$$\beta_3 = -18 \lim_{\tilde{m}_{\sigma} \rightarrow 0} \tilde{m}_{\sigma}^2 \text{Sl}(\tilde{m}_{\sigma}^2) \simeq 1.645 \tag{110}$$

$$\beta_3' = -2 \lim_{\tilde{m}_{\sigma} \rightarrow 0} \tilde{m}_{\sigma}^2 [\text{Pl}(\tilde{m}_{\sigma}^2) + \text{Ml}(\tilde{m}_{\sigma}^2)] \simeq 1.805 \tag{111}$$

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